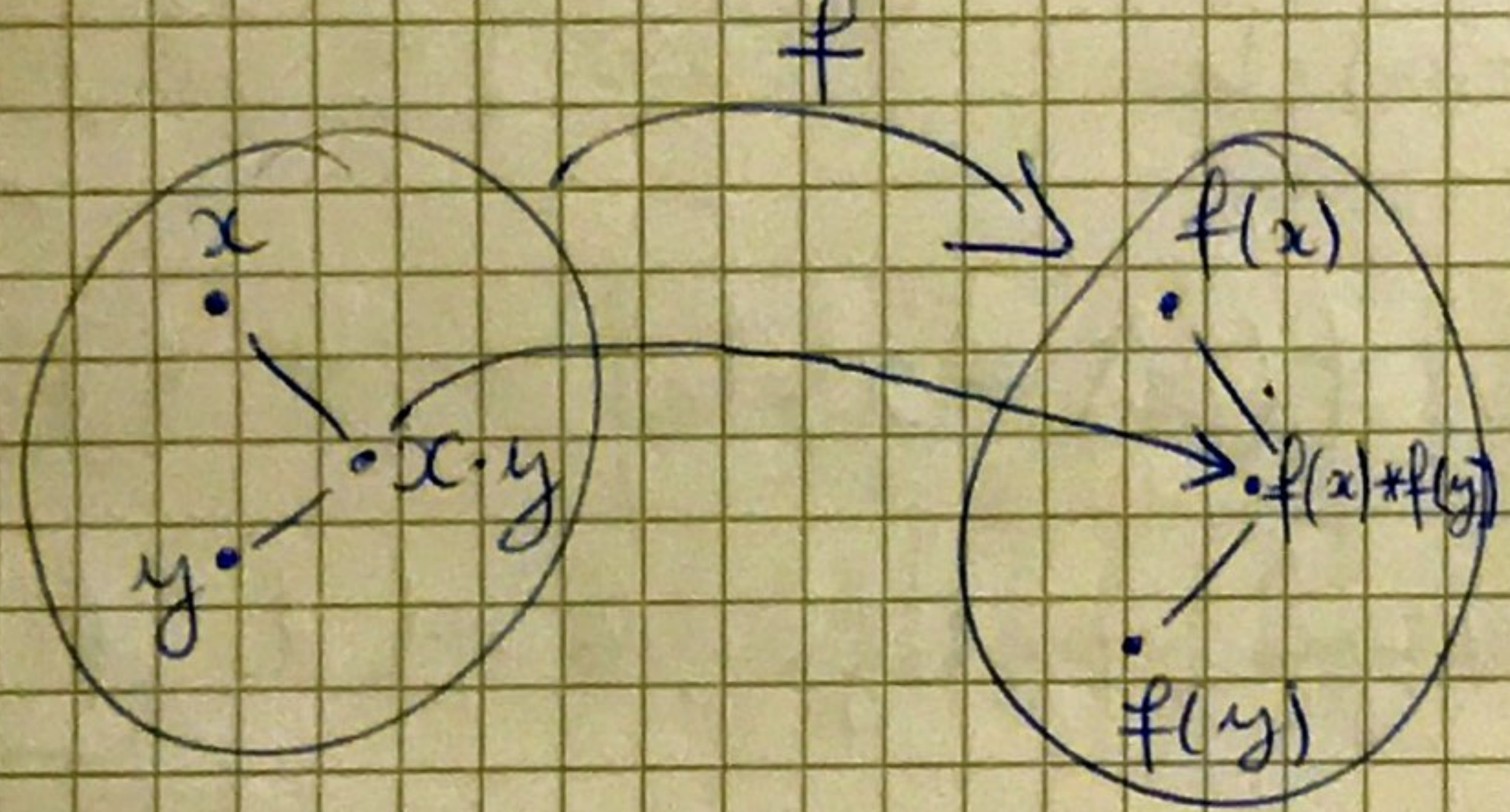
Def. $(G, \cdot)$, $(S, *)$ − grupe

Za f kažemo da je homomorfizam (grupa G i S) ako

$$\forall x, y \in G \quad f(x \cdot y) = f(x) * f(y)$$



Injektivni homomorfizam je monomorfizam.
Surjektivni homomorfizam je epimorfizam.
Bijektivni homomorfizam je izomorfizam.

Ako je $G = S$ onda se koristi naziv automorfizam.

7. Odrediti sve automorfizme aditivne grupe $(\mathbb{Z}, +)$

$$f : \mathbb{Z} \to \mathbb{Z}$$

$$f(x+y) = f(x) + f(y)$$

$$f(0+0) = f(0) + f(0)$$

$f(0) = f(0) + f(0) \Rightarrow f(0) = 0$

$f(1) = k, \ k \in \mathbb{Z}$

$n \in \mathbb{N}$

$f(n) = f(\underbrace{1+1+\dots+1}_{n}) = \underbrace{f(1)+\dots+f(1)}_{n} = n f(1)$

$f(n) = n \cdot k$

$f(-n) = ?$

$f(n) + f(-n) = f(\underbrace{n+(-n)}) = 0$

$f(-n) = -f(n) = (-n)^{\circ} \cdot k$

$\Rightarrow f$ je oblika $f(x) = x \cdot k$

Kako je f bijekcija, postoji b t.d. $\underbrace{f(b) = 1}_{b \cdot k = 1}$

$1^\circ \quad k = b = 1$

$2^\circ \quad k = b = -1$

$f(x) = kx \begin{cases} \boxed{f(x) = x} \\ \boxed{f(x) = -x} \end{cases}$

Osnovna teorema o homomorfizmu grupa:

$f : G \to G'$ – homomorfizam grupa $(G, \circ)$ i $(G', *)$

Tada je $\operatorname{Ker} f \trianglelefteq G$. Važi i obrnuto, ako je $H \trianglelefteq G$

tada postoji faktor grupa $(G/_H, \odot)$ i epimorfizam

$\pi : G \to G/_H$ t.d. $\operatorname{Ker}\pi = H$ i važi

$G/_{\operatorname{Ker} f} \cong \operatorname{Im} f$ $\qquad C_x \odot C_y = C_{x \cdot y}$

(8.) Dokazati da je $G/_{GL(1,\mathbb{R})} \cong (\mathbb{R}^+, \cdot)$

$GL(1,\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid |\det A| = 1 \}$ $\qquad$ G – reg. mat. u $\mathbb{R}^{n\times n}$

$f : G \to \mathbb{R}^+$

$f(A) = |\det A|$

$$f(A\cdot B) = |\det(A\cdot B)| = |\det A \cdot \det B| =$$
$$= |\det A| \cdot |\det B| = f(A)\cdot f(B) \implies$$

$\implies f$ je homomorfizam

$\operatorname{Ker} f = \{ A \in G \mid |\det A| = 1 \} = GL(1,\mathbb{R})$

Neka je $y \in \mathbb{R}^+$.

? $\exists A \quad |\det A| = y$

$$\left| \begin{pmatrix} y & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \right| = y$$

$\mathrm{Im} f = \mathbb{R}^+$

$\overset{T.}{\Longrightarrow} G/_{GL(1,\mathbb{R})} \cong (\mathbb{R}^+, \cdot)$

9. $G = (\mathbb{R} \setminus \{0\}, \cdot)$

$S = \{x \in \mathbb{R} \mid x \neq -1\}$

$(S, *)$

$x * y = x + y + xy$

Dokazati da je $f(x) = x - 1$ izomorfizam grupa G i S.

$? \; \forall x, y \quad f(x \cdot y) = f(x) * f(y)$

$L = f(x \cdot y) = xy - 1$

$$\begin{aligned} D = f(x) * f(y) &= (x-1) * (y-1) = \\ &= (x-1) + (y-1) + (x-1)(y-1) = \\ &= \cancel{x} - 1 + \cancel{y} \cancel{-1} + xy \cancel{-x} \cancel{-y} \cancel{+1} = \\ &= xy - 1 = L \end{aligned}$$

$\Rightarrow f$ je homomorfizam

$f(x) = x - 1$

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{-1\}$

$f(x) = f(y)$

$x - 1 = y - 1 \Rightarrow x = y \Rightarrow f$ je inj. (monomorfizam)

$y \in \mathbb{R} \setminus \{-1\}$

za $\frac{x}{2} = \frac{y+1}{1}$ je $f(x) = y+1-1 = y \Rightarrow$

$\Rightarrow f$ je surj. (epimorfizam)

$\Rightarrow f$ je izomorfizam

(10.) G – grupa

$C_G(a) = \{ x \in G \mid x * a = a * x \}$

$C_G(a) \leq G$

$x, y \in C_G(a)$

$x * a = a * x$

$y * a = a * y$

$y * a = a * y \,/\, y^{-1}$

$y^{-1} \backslash\, y * a * y^{-1} = a$

$a * y^{-1} = y^{-1} * a$

$\Downarrow$

$y^{-1} \in C_G(a)$

$x * \underbrace{y^{-1} * a} = \underbrace{x * a} * y^{-1} =$

$= a * x * y^{-1} \Rightarrow$

$\Rightarrow x * y^{-1} \in C_G(a) \Rightarrow$

$\Rightarrow C_G(a) \leq G$

2.od $a \in G$ fix, $(G, *)$ csoport

Dök: $f(x) = a * x * a^{-1}$ homomorfizmus $f: G \to G$

Bi $f(x) * f(y) = (a * x * \underbrace{a^{-1}) * (a}_{e} * y * a^{-1})$

$= a * x * y * a^{-1} = f(x * y))$

$\mathrm{Ker} f = ?$ $\quad f(x) = e$

$a * x * a^{-1} = e \ / a *$

$a^{-1} / \ a * x = a$

$\Rightarrow \boxed{x = e}$ $\quad \Rightarrow \mathrm{Ker} f = \{e\}$

$y \in G$ $\quad f(x) = y = a * x * a^{-1}$

$a^{-1} / \ a * x * a^{-1} = y \ / a$

$x = a^{-1} y a$

$f(x) = f(a^{-1} y a) = a(a^{-1} y a) a^{-1} = y$

$\mathrm{Im} f = G$

$\boxed{G/\{e\} \cong G}$

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(Zad) G grupa svih $f: \mathbb{R} \to \mathbb{R}$ sa operacijom $(f+g)(x) = f(x)+g(x)$. Neka je H podskup koji se sastoji od svih f t.d. $f(0)=0$.

Dok: $G/H \cong (\mathbb{R},+)$

$G/\ker f \cong \operatorname{Im} f$

(R)

$$\psi : G \to \mathbb{R}$$

$$\psi(f) = f(0)$$

$$\psi(f+g) = (f+g)(0) = f(0)+g(0) = \psi(f)+\psi(g)$$

$$\ker\psi = \{f \in G \mid \psi(f)=0\} = \{f \in G \mid f(0)=0\} = H$$

$$\operatorname{Im}\psi = \{y \in \mathbb{R} \mid \exists f \in G \ \ \psi(f)=y\} = \{y \in \mathbb{R} \mid \exists f \in G \ \ f(0)=y\} = \mathbb{R}$$

jer npr za $\tilde{f}(x) = y$, $\forall x$
const.

$\Rightarrow \tilde{f}(0) = y$

$$\Rightarrow \boxed{G/H \cong \mathbb{R}}$$

Simetrične grupe

$$X \neq \emptyset, \quad |X| = n, \quad X = \{1, 2, \dots, n\}$$

$$\pi : X \xrightarrow{\text{bij.}} X$$

$$\pi : \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$(S_n, \circ)$ – grupa

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Ciklusi

$$a_1 \to a_2 \to a_3 \to \cdots \to a_{k-1} \to a_k \to a_1$$

$$(a_1 a_2 a_3 \dots a_k)$$

Lema.
Svaka permutacija se može zapisati kao proizvod nezavisnih (disjunktnih) ciklusa.

Primjer. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 5 & 7 & 3 & 4 & 6 & 8 \end{pmatrix}$

$(1\,2)\circ(3\,5)\circ(4\,7\,6)\circ(8)$ $\quad (*)$

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & i & \cdots & j & \cdots & n \\ 1 & 2 & 3 & \cdots & j & \cdots & i & \cdots & n \end{pmatrix} = (i\,j) - \text{transpozicija}$$

Napomena.

U $(*)$ nije bitan redoslijed nezavisnih ciklusa.

Znak permutacije:

$$\{\underset{\text{neparne}}{-1}, \underset{\text{parne}}{1}\}$$

$$\pi : \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

$$\operatorname{sgn} \pi = \prod_{i<j} \frac{\pi(j) - \pi(i)}{j - i}$$

1. Naći znak permutacije:

a) $\pi : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\,3)(2)$

b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\,3)(2\,4)$

a) $\operatorname{sgn} \pi = \frac{2-3}{2-1} \cdot \frac{1-2}{3-2} \cdot \frac{1-3}{3-1} = -1$

b) $\operatorname{sgn}\pi = \overset{+}{\frac{4-3}{2-1}} \cdot \overset{-}{\frac{1-4}{3-2}} \cdot \overset{+}{\frac{2-1}{4-3}} \cdot \overset{-}{\frac{1-3}{3-1}} \cdot \overset{-}{\frac{2-4}{4-2}} \cdot \overset{-}{\frac{2-3}{4-1}} =$

$= +1$

<u>II način.</u>

Inverzija:

a)
$$\left.\begin{array}{ccc} 1 & 2 & - \\ 1 & 3 & - \\ 2 & 3 & - \end{array}\right\} -1$$

b)
$$\left.\begin{array}{ccc} 1 & 2 & + \\ 1 & 3 & - \\ 1 & 4 & - \\ 2 & 3 & - \\ 2 & 4 & - \\ 3 & 4 & + \end{array}\right\} +1$$